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Evaluating the Helmholtz Integral:

Part 1 - Basic Theory

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In the Kirchhoff theory of physical optics the field scattered by a reflecting surface is represented by a Helmholtz integral, which is a certain double integral evaluated over the surface. New results are presented which show how the Helmholtz integral has a closed form representation as a line integral evaluated over the boundary of the surface. Also, in the standard far-field approximation to the Helmholtz integral, the scattered field can be represented in a form which does not involve any integrations at all. These results are very attractive from

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the standpoint of economy of calculation and appear to be well suited to the problem of estimating the field scattered by a large number of metallic plates, especially when the effects of multiple scatterings are of concern.

CONTENTS

EVALUATING THE HELMHOLTZ INTEGRAL: PART 1 — BASIC THEORY

1. INTRODUCTION

The research described in this report was motivated by the problem of computing the electromagnetic field scattered by a large number of metallic reflectors. This problem arises in the study of the proposed new doppler Microwave Landing System, and our main concern is to determine the effects of multiple scatterings between the scattering objects. As a first approximation to a solution of this problem it will be assumed that the collection of reflectors consists mainly of flat metallic plates of arbitrary shape, and the Kirchhoff theory of diffraction will be used to estimate the fields scattered by the plates.

In the Kirchhoff theory of diffraction, the field scattered by a surface is represented by a Helmholtz integral, which is a rather complicated surface (double) integral evaluated over the surface, and in Sec. 2 we shall present some new results showing how the Helmholtz integral has a closed form representation as a line integral evaluated over the boundary of the reflecting surface.

In the standard far-field approximation to the Kirchhoff theory we can even do more: If the reflecting surface is bounded by straight lines, then the far-field approximation can be reduced to the sum of a number of terms ΣJ_n, each term J_n being merely a certain complex quantity evaluated at the nth vertex of the reflector. In other words, for flat polygonal reflectors the usual far-field approximation reduces to a form which requires no integrations at all, and hence is extremely attractive from the standpoint of economy of calculation. These results will be presented in Sec. 4.

The results of Sec. 2 might also be applied to obtain closed form expressions for the "exact" Helmholtz integral (rather than the far-field approximation) which involve no integrations. This possibility will be the subject of future investigations.

Throughout this report it will be assumed that the radiation incident on each reflector is either a plane wave or a spherical wave; *however, our methods generalize to other types of incident radiation, and only require that the radiation satisfy the wave equation, and that the scattering objects be finitely extended.*

2. REDUCTION OF THE HELMHOLTZ INTEGRAL TO A LINE INTEGRAL

Notation and Definitions

S is an open surface in euclidean 3-space which, physically, will correspond to a metallic reflector or an aperture in an opaque screen. Unless otherwise stipulated, it will always be assumed that S is finite in extent.

$\partial S \equiv$ boundary of S.

$\vec{\eta} \equiv$ unit normal to S, and ∂S is always oriented so that a point moving in a positive direction around ∂S appears to move in a counterclockwise direction when $\vec{\eta}$ is pointing towards the observer.

The Helmholtz integral is given by the right-hand side of the relation

$$4\pi u_P = \int_S \left\{ u \text{ grad } H - H \text{ grad } u \right\} \cdot \vec{\eta}\, dA \tag{2.1}$$

where u_P is the (purported) value of the scattered field at a fixed "field point" P, and $H = [\exp(ikr)]/r$ where $k = 2\pi/\lambda$ is the wave number and where $\vec{r}$ is the vector drawn from P to a variable point in space.

Let $\vec{\xi}$ be a unit vector, which in our applications will correspond to the direction in which a wave front is moving as it passes the field point P and let ℓ be the axis which passes through P and is parallel to $\vec{\xi}$. Let φ be the angular coordinate which corresponds to a rotation around ℓ, so that if a special xyz coordinate system is chosen with $\vec{e}_z = \vec{\xi}$, we have

$$d\varphi = (xdy - ydx)/(x^2 + y^2). \tag{2.2}$$

Equivalently, if F is a function, the line integral $\int_{\partial S} F d\varphi$ can be written

$$\int_{\partial S} F d\varphi = \int_{\partial S} F \frac{(\vec{\xi} \times \vec{r}) \cdot \frac{d\vec{r}}{dt}}{|\vec{\xi} \times \vec{r}|^2} dt \tag{2.3}$$

where $\vec{r} = \vec{r}(t)$ is any parametric representation of ∂S. Then we shall say that P is an I-point, a B-point or an S-point according as ∂S winds around ℓ, ℓ intersects ∂S, or ℓ falls outside of ∂S (see Fig. 1.). Analytically, the condition that P be an I-point is that $\int_{\partial S} d\varphi = \pm 2\pi$ (the sign depending on the orientation of ∂S), and the condition that P be an S-point is $\int_{\partial S} d\varphi = 0$. If P is a B-point, the form $\int_{\partial S} d\varphi$ is indeterminate. Physically, the sets of I-points, B-Points, and S-points correspond (respectively) to the geometrical optics illuminated zone, shadow boundary zone, and shadow zone *as they are predicted by the Kirchhoff method.* (See Sec. 3 for further discussion.)

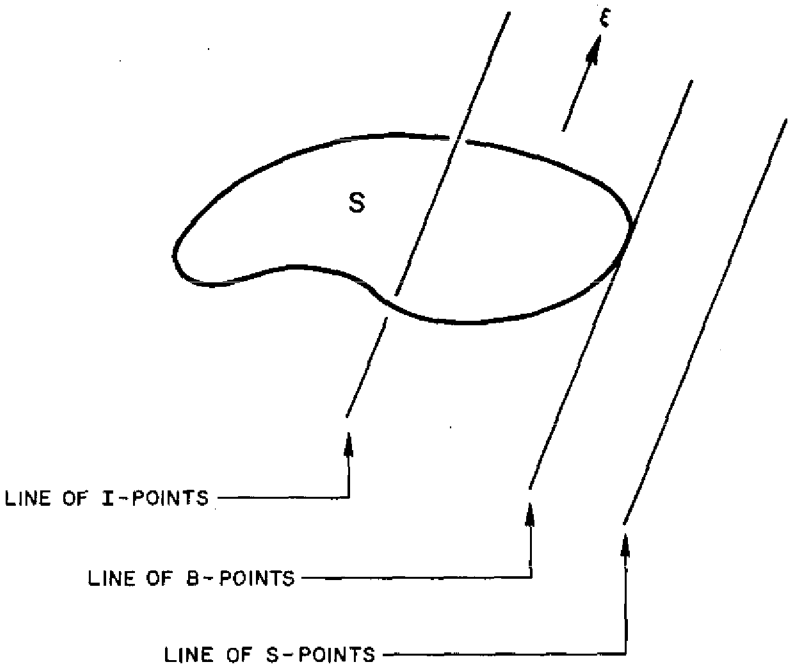

Figure 1

Preliminary Remarks

Since the reduction of a surface integral to a line integral is in one sense trivial, we shall interrupt our presentation at this point to discuss what is at issue here.

From Stokes' (or Green's) Theorem we have

$$\int_{\partial S} pdx + qdy = \int_S \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy.$$

Hence in order to reduce a surface integral $\int_S fdxdy$ to a line integral, we only have to solve the equation

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = f \qquad (*)$$

for p and q. Such a solution is always possible provided only that S be simply connected. For example, if S is convex, we can set $p = 0$, so that we only have to obtain q by integrating $\partial q/\partial x = f$. (If S is not convex, there is a difficulty in defining the limits of integration.) However, such a procedure is unsatisfactory from two points of view: For the purposes of computation we have accomplished nothing, since one integration is required to solve (*), and a second integration is required to evaluate the line integral. In other words, all we have done is to express the surface integral as an iterated double integral. Also, from the standpoint of theory we have gained nothing since none of the infinite number of solutions to (*) appear to have any geometric or physical content.

What we want therefore is a representation of the Helmholtz integral as a line integral which is in *closed form*, i.e., a representation as a line integral $\int_{\partial S} Fd\varphi$ in which the integrand F does not depend on S and which is valid for any member of a class of incident radiation which is given explicitly in a functional form $u = u(\vec{r}, t_1, \ldots, t_n)$ involving a finite number of parameters $t_1, \ldots, t_n$.

Statement of Results for the Plane and Spherical Wave Case

The following propositions will illustrate results of the kind indicated. We should emphasize that the relations (2.4) and (2.5) are derived by purely mathematical processes, involve no approximations, and hence are perfectly exact.

Proposition 1. *(Plane Wave Case). Let all vectors and angles be as shown in Fig. 2, and let* $u(\vec{r}) = \exp(ik\vec{r}\cdot\vec{\xi})$.

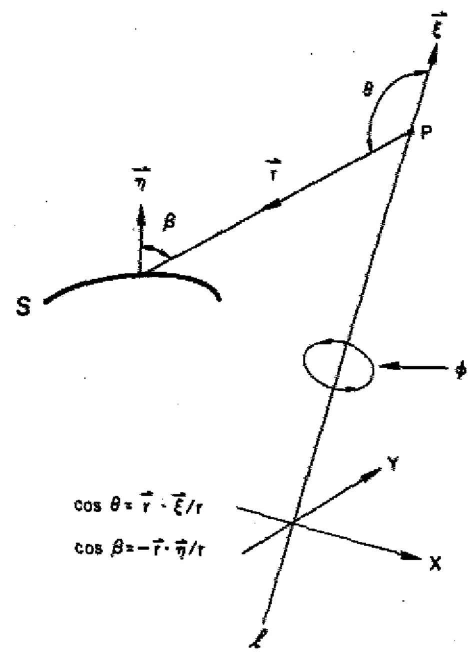

Fig. 2 — Plane wave case

Then

$$\int_S \{u \text{ grad } H - H \text{ grad } u\} \cdot \vec{\eta}\, dA = c - \int_{\partial S} (1 - \cos\theta) \exp[ikr(1+\cos\theta)]\, d\varphi \tag{2.4}$$

where $c = \begin{cases} 0 \text{ if } P \text{ is an S-point} \\ 4\pi \text{ if } P \text{ is an I-point.} \end{cases}$

Proposition 2. *(Spherical Wave Case). Let all vectors and angles be as is shown in Fig. 3, and let P_o be the source of a spherical wave $u(\vec{r}_o) = (1/r_o)\exp(ikr_o)$ where $\vec{r}_o$ is the vector drawn from P_o to a variable point in space.*

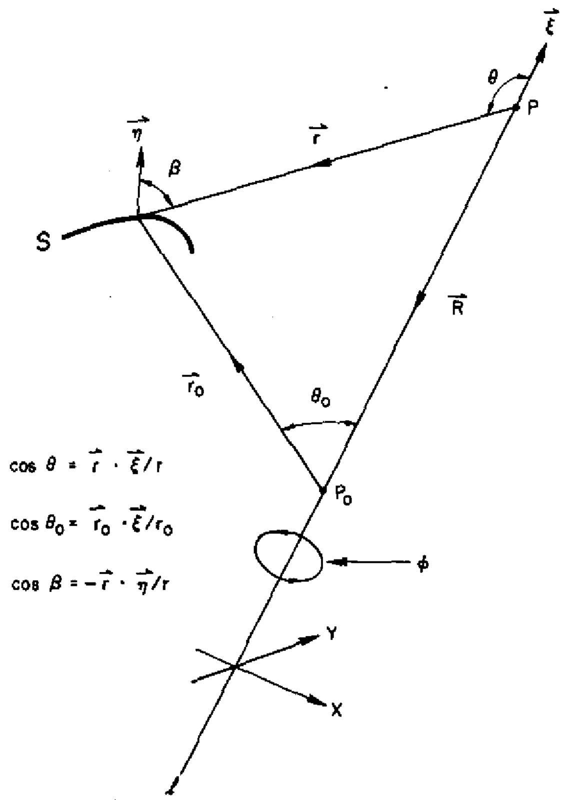

Fig. 3 — Spherical wave case

Then

$$\int_S \left\{ u \text{ grad } H - H \text{ grad } u \right\} \cdot \vec{\eta}\, dA = \frac{e^{ikR}}{R} \left\{ c - \int_{\partial S} \left[1 - \cos(\theta - \theta_o)\right] \exp[ik(r_o + r - R)]\, d\varphi \right\} \tag{2.5}$$

where $c = 0$ or 4π as before.

Remark — In Proposition 1 the "incident" field u is always normalized so that the phase is zero at the field point. To obtain a result which preserves the proper phase relations as the field point P is varied one should multiply the right-hand side of (2.4) by $\exp(ik\vec{R}\cdot\vec{\xi})$ where for each field point P, $\vec{R}$ is the position vector of P drawn from some fixed origin.

3. GENERAL DISCUSSION

The derivations of formulas (2.4) and (2.5), together with a discussion of their physical significance, is given in Ref. 1, and the results of detailed numerical calculations will be given in subsequent reports. In this section we shall discuss how these formulas are to be used in scattering problems.

A. As they stand, the formulas are immediately applicable to the case where S is an aperture in an opaque screen. The Kirchhoff method consists in setting each component u of the incident field in the left-hand side of (2.4) and (2.5) equal to the value

$$u(x) = \begin{cases} \text{value of unperturbed field for } x \text{ in } S \\ 0 \text{ for } x \text{ in planar set complementary to } S. \end{cases}$$

The case of flat planar reflectors is treated in the same way, with S being regarded as an aperture, except that, in the plane wave case, the incident ray $\vec{\xi}$ is replaced by the reflected ray

$$\vec{\xi}_r = \vec{\xi} - 2(\vec{\xi}\cdot\vec{\eta})\vec{\eta}; \tag{3.1}$$

while in the spherical wave case, the source P_o is replaced with its "image" source located symmetrically on the opposite side of the plane. More generally, for the case of curved reflectors, S is taken to be a certain subdomain (determined by geometrical optics) of a plane approximating an emerging wave front.

B. We shall now justify the use of our I-, S-, and B-point terminology. If S is a plane surface, then, from geometry, our I, S, and B zones obviously correspond to the geometric optics illuminated, shadow, and shadow, and shadow boundary zones, respectively. Suppose now that S is curved. Then our terminology is still justified because our I and S zones correspond to the geometric optics field *as it is predicted by the Kirchhoff method.* The reason for this is that the line integrals in (2.4) and (2.5) converge to zero as $k \to \infty$ except in certain rare pathological cases. For definiteness, consider the plane wave case. Then the only way in which the line integral could fail to converge to zero as $k \to \infty$ is if the quantity $r(1+\cos\theta)$ remained constant over some nonzero segment of ∂S. (This happens, for example, when one computed the field along the axis of a disc in the case when $\vec{\xi}$ is normal to the disc.)

C. Although the line integrals in (2.4) and (2.5) appear to be very well suited for the purposes of machine calculation, one cannot expect to be able to obtain closed form expressions for these integrals except in the simplest cases. However, one can apply the Principle of Stationary Phase to obtain closed form expressions for the line integrals which are asymptotically valid as $k \to \infty$. In applying this method one should take care to express all variables (including φ) in terms of a parameter t with the property that every t-value corresponds to only one point of ∂S. The parameters r, θ, and φ do *not* have this property at S-points.

From the standpoint of theory the Principle of Stationary Phase is interesting because it is suggestive of Fermat's Principle and J. B. Keller's Geometric Theory of Diffraction [2-4]. Consider, for example, the spherical wave case (2.5), and recall that R is constant (since P_o and P are fixed during the integration). Then according to this principle an asymptotically valid expression is obtained which contains only a finite number of terms, each term corresponding to a value of t for which $r_o + r = r_o\ (t) + r(t)$ is stationary. There exist at least two such terms, corresponding to points on ∂S at which $r_o + r$ attains its minimum and maximum values.

Recall that in the case of flat plate reflectors (cf. *(A)* above) the source point P_o is actually the "image" source. From this remark it is easily seen that the c term in (2.5) corresponds to the specular component of reflection.

The plane wave case is entirely similar. Let $\vec{r} = \vec{PQ}$, where Q is a variable point on ∂S. Then $r(1+\cos\theta) = -a + r +$ distance from Q to a fixed but arbitrary wave front below S, where a is a constant (dependent on the wave front selected).

Both geometrical optics effects and edge effects contained in the Helmholtz integral have been previously observed. Keller [4] obtained similar results by applying the two-dimensional Principle of Stationary Phase to the double integral (2.1).

D. The formulas (2.4) and (2.5) are defintely not valid when S is infinitely extended. The reason for this appears to be that the derivation of these formulas requires repeated application of Stokes' Theorem, which is only valid on finitely extended surfaces. In particular, in the case of the infinite half-plane, the integrals converge but yield incorrect results.

4. THE FAR-FIELD APPROXIMATION

This section is concerned with the far-field approximation to the field scattered by a flat plate whose boundary consists of straight-line segments. It will be shown how the standard far-field approximation reduces to a form which does not involve any integrations. The derivation given below is independent of the results of Sec. 2, and only involves the use of Stokes' Theorem and the Divergence Theorem.

The formulas given below apply to the case when S is an aperture in an opaque screen. To obtain corresponding results for the case when S is a reflector, one must reflect the incident field ray $\vec{\xi}$ as is indicated in Eq. (3.1).

We consider a (scalar) plane wave u of unit field strength incident on a plane aperture S. The first thing we wish to establish is that there is no loss of generality in assuming that the phase illumination is constant on S. For if $\vec{\xi}$ is not normal to S we can apply the Helmholtz formula (2.1) to the surface S' which is the surface obtained by projecting S onto a plane perpendicular to $\vec{\xi}$, keeping a vertex of S fixed.

Proof. Since div $\{u \text{ grad } H - H \text{ grad } u\} = 0$, we have (Divergence Theorem)

$$\left\{ -\int_S + \int_{S'} + \int_{S''} \right\} (u \text{ grad } H = H \text{ grad } u) \cdot \vec{\eta} dA = 0$$

where S'' is the surface which consists of those "sides" which, together with S and S', form a closed surface S_o. The *minus* sign appears only in the first term because the normal $\vec{\eta}$ points outward at S' and S'' and inward to S_o at S.

But $\int_{S''} = 0$ since the sides of S'' are parallel to $\vec{\xi}$.

Hence $\quad \int_S = \int_{S'}$.

We now assume that the incident radiation has unit strength and constant phase illumination on the plane aperture S.

Let x_1, x_2, x_3 be standard rectangular coordinates, and let S lie in the $x_1 x_2$ plane. Let R, θ, φ be spherical coordinates, with

$$x_1 = R \sin\theta \cos\varphi ,$$

$$x_2 = R \sin\theta \sin\varphi ,$$

$$x_3 = R \cos\theta .$$

If the origin of these coordinate systems is made to lie in the interior of S, then the standard far-field approximation for $u(P)$ is given by

$$u(P) = -\frac{ik(1 + \cos\theta)e^{ikR}}{4\pi R} \int_S e^{-ik\vec{x}\cdot\vec{v}} \, dx_1 dx_2 \tag{4.1}$$

where

$$\begin{aligned} \vec{x} &= [x_1, x_2] \\ \vec{v} &= [v_1, v_2] = [\sin\theta \cos\varphi, \sin\theta \sin\varphi] \end{aligned} \tag{4.1a}$$

(cf. Ref. 5, p. 173. Some authors commonly write $-i$ where we write $+i$.)

The main point of our derivation is contained in the following lemma.

Lemma. *Let $\vec{w} = [w_1, w_2]$ be a constant vector. Then*

$$\int_S e^{i\vec{x}\cdot\vec{w}} dx_1 dx_2 = \frac{i}{|\vec{w}|^2} \int_{\partial S} e^{i\vec{x}\cdot\vec{w}} (w_2 dx_1 - w_1 dx_2) . \tag{4.2}$$

Proof. Apply Stokes' Theorem (or Green's Theorem) to the right-hand side of (4.2).

We now apply (4.2) to (4.1), with $\vec{w} = -k\vec{v}$. Let $\vec{x} = \vec{x}(t)$ be any parametric representation of ∂S. We get

$$-u(P) = \frac{1 + \cos\theta}{\sin\theta} \cdot \frac{e^{ikR}}{4\pi R} \cdot \int_{\partial S} e^{-i(k \sin\theta)\vec{x}\cdot\vec{e}} \left(\vec{e}^{\,*} \cdot \frac{d\vec{x}}{dt}\right) dt \tag{4.3}$$

where

$$\vec{e} = [\cos\varphi, \sin\varphi]$$

$$\vec{e}^{\,*} = [\sin\varphi, -\cos\varphi].$$

Remark. If $\theta = 0$, then from (4.1) we get

$$u(P) = \frac{-ike^{ikR}}{2\pi R} \quad A$$

where A = area of S.

Finally, we want to reduce the right-hand side of (4.3) to a form which does not involve any integrals. Let S be a plane polygon with N vertices $\vec{a}_1, \ldots, \vec{a}_N$, each vertex $\vec{a}_n$ being a 2-vector in the x_1x_2 plane. Set $\vec{a}_{N+1} = \vec{a}_1$, and let $\vec{\Delta a}_n = \vec{a}_{n+1} - \vec{a}_n$. Then for $\sin\theta \neq 0$, we have

$$-u(P) = \frac{1 + \cos\theta}{\sin\theta} \cdot \frac{e^{ikR}}{4\pi R} \cdot (J_1 + \ldots + J_N), \tag{4.4}$$

where, for $1 \leqslant n \leqslant N$,

$$J_n = (\vec{e}^{\,*} \cdot \vec{\Delta a}_n) \frac{\sin\left\{\dfrac{k \sin\theta}{2} (\vec{e}\cdot\vec{\Delta a}_n)\right\}}{\left\{\dfrac{k \sin\theta}{2} (\vec{e}\cdot\vec{\Delta a}_n)\right\}} \exp\left[-i(k \sin\theta) \left(e \cdot \frac{\vec{a}_n + \vec{a}_{n+1}}{2}\right)\right].$$

Proof. We have to show that the expression for J_n in (4.4) is the contribution of the nth side of the polygon S to the integral in (4.3).

A parametric representation of the nth side of S is given by

$$\vec{x}(t) = (1-t)\vec{a}_n + t\vec{a}_{n+1}, \quad 0 \leqslant t \leqslant 1,$$

so that

$$\frac{d\vec{x}}{dt} = \vec{\Delta a}_n.$$

Substituting these relations into the integrand in (4.3), we are led to an integration of the type $\int_0^1 \exp(ibt)dt$ where b is a certain constant. Performing all the arithmetic we ultimately arrive at the expression for J_n in (4.4).

5. SUMMARY AND CONCLUSIONS

The Helmholtz integral which occurs in the Kirchhoff Theory of Diffraction (Eq. (2.1)) has been reduced to a line integral (Eqs. (2.4) and (2.5)). For the case of a plane wave incident on a flat polygonal plate, the standard far-field approximation to the scattered field has been further reduced to a form which involves no integrations at all.

From the standpoint of theory these line integral representations of the Helmholtz integral are interesting because they show very clearly how the Kirchhoff theory predicts certain geometrical optics and edge effects, and because they suggest certain ways in which the Kirchhoff theory might be improved. From the standpoint of practice the line integral representations have value because they result in great economy of calculation, and we shall conclude this report with a discussion of this effect.

If N sample points are required to evaluate the line integral of a function over the boundary of a domain, then N^2 points would be required to evaluate the surface (double) integral of the function over the domain with the same degree of precision. (More generally, the error in numerically integrating a function over a unit d-dimensional hypercube varies as $N^{-A/d}$ where A is a constant involving bounds on the derivatives of the function.) For example, consider a rather small scattering surface which has a circumference of about five wavelengths. Typically, ten sample points per wavelength might be required to numerically evaluate the line integral with an acceptable degree of precision, so that $N = 50$ points would be required for the line integral and $N^2 = 2500$ points would be required for the corresponding Helmholtz (double) integral.

Now in the computer analysis of the doppler Microwave Landing System, the computation of the scattered field at a single point will require computing the complicated interactions between dozens of scattering objects. Moreover, we shall be interested in seeing how the spectrum of the received signal varies with time as an aircraft approaches a landing, and this might require the computation of the scattered field at thousands of points along each aircraft trajectory. Also, the trajectories, the geometrical arrangements of the scatterers, and the parameters of the system will be varied. Hence, an increase in the speed of computation by a factor of from 10 to 100 will have a considerable effect on our ability to use a computer to analyze the performance of the doppler Microwave Landing System.

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